

Final Exam at 13, MTH 213, Fall 2018

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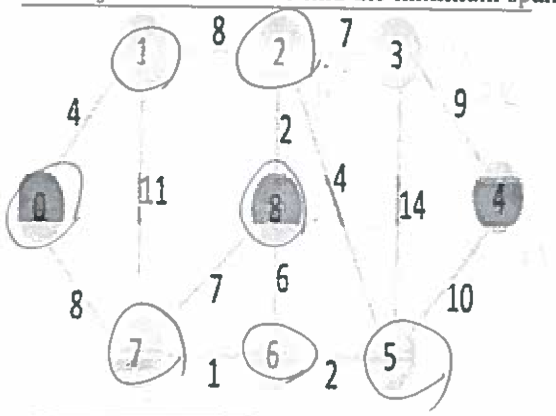
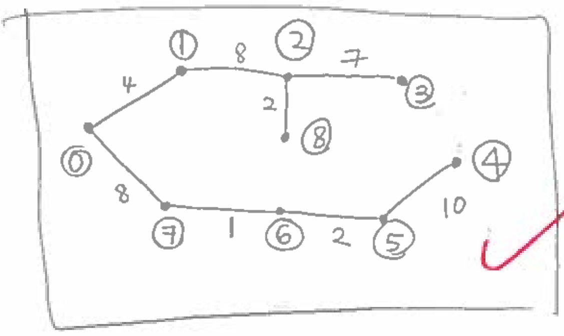
$Score = \frac{77}{80}$

QUESTION 1. (8 points)

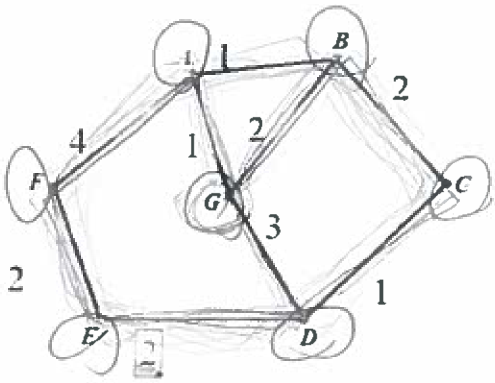
Use Dijkstra's method to find the minimum spanning tree of the below graph (you may start from vertex 0).



vertex	0	1	2	3	4	5	6	7	8
0	$\boxed{0}$	4^0	∞	∞	∞	∞	∞	8^0	∞
1		$\boxed{4^0}$	12^1	∞	∞	∞	∞	8^0	∞
7			12^1	∞	∞	∞	9^7	$\boxed{8^0}$	15^7
6			12^1	∞	∞	11^6	$\boxed{9^7}$		15^7
5			12^1	25^5	21^5	$\boxed{11^6}$			15^7
2			$\boxed{12^1}$	19^2	21^5				14^2
8				19^2	21^5				$\boxed{14^2}$
3				$\boxed{19^2}$	21^5				
4					$\boxed{21^5}$				



QUESTION 2. (5 points)



A salesman is located at G. He wants to visit each block (each vertex) exactly once and then return to G.

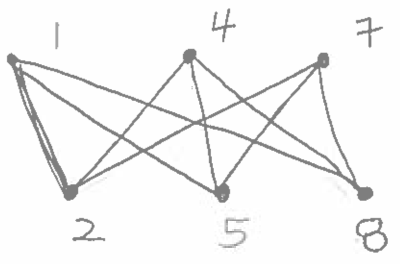
1) Find all possible Hamiltonian cycle.

- $G \xrightarrow{1} A \xrightarrow{4} F \xrightarrow{2} E \xrightarrow{2} D \xrightarrow{1} C \xrightarrow{2} B \xrightarrow{2} G$ weight: 14
- $G \xrightarrow{2} B \xrightarrow{2} C \xrightarrow{1} D \xrightarrow{2} E \xrightarrow{2} F \xrightarrow{4} A \xrightarrow{1} G$ weight: 14

2) Find the Hamiltonian cycle with minimum weight. G – A – F – E – D – C – B – G

QUESTION 3. (6 points) Let $V = \{1, 2, 4, 5, 7, 8\}$. Two vertices in V, say a, b, are connected by an edge if and only if $a + b = 3c$ for some $c \in N^*$.

a) Draw such graph.



b) Is the graph a complete bipartite graph? if it is a $K_{n,m}$, then find n and m.

Yes, the graph is a complete bipartite graph. $n = m = 3$.

c) Find the diameter of the graph.

diameter is 2.

d) Is the graph an Eulerian? If yes, then find such Eulerian circuit.

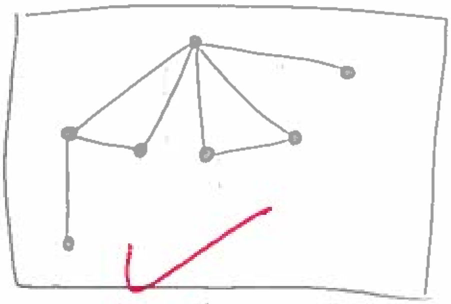
No, the graph is not Eulerian. ($\because$ deg(4) = 3)

e) Is the graph Hamiltonian? If yes, then find such Hamiltonian cycle.

yes; 1-2-7-8-4-5-1.

QUESTION 4. (4 points) Is the sequence 5, 3, 2, 2, 2, 1, 1 graphical (i.e., is there a graph so that the vertices have the given degrees)? If yes draw such graph.

(i) $\boxed{5}$, 3, 2, 2, 2, 1, 1
(ii) 2, 1, 1, 1, 0, 1
(iii) $\boxed{2}$, 1, 1, 1, 1, 0
(iv) 0, 0, 1, 1, 0
(v) $\boxed{1}$ 1, 0, 0, 0
(vi) 0, 0, 0, 0
↳ yes, there is a graph with the given degrees = 5, 3, 2, 2, 2, 1, 1



QUESTION 5. (6 points)) Consider the following code.

For $k = 3$ to $(n^3 + 5)$ → $n^3+5-3+1 = \boxed{n^3+3}$
$S = k^3 + 3 * k + 4$ → 5 → outer
For $i = 2$ to $(k + 9)$ → $k+9-2+1 = \boxed{k+8}$
$L = i^2 + 7 * i + 2$ → 4 → inner
next i
next k

(i) Find the exact number of addition, subtraction, multiplication that the code executed.

# of times outer loop executed : $\boxed{n^3+3}$

outerloop k=	k=3	$k=n^3+5$
innerloop i=	# of times innerloop executed : 11 # of operations: 44	# of times innerloop executed : $n^3+5+8 = n^3+13$ # of operations : $4(n^3+13) = 4n^3+52$

# of operations outer loop : $5(n^3+3) = 5n^3+15$

$\therefore$ exact number of operations $= \dfrac{(44+4n^3+52)(n^3+3)}{2} + (5n^3+15)$

(ii) Find the complexity of the code.

O (code) $= n^6$

QUESTION 6. (4 points) $A = \{4, 6, 7, 8, 9, 11, 13\}$ and let $B = P(A)$ (i.e., B is the power set of A).

(a) Find $|B|$.

$|A| = 7$

$|B| = |P(A)| = 2^7 = 128$

(b) Define " $=$ " on B such that $\forall a, b \in B$, $a" = "b$ if and only if $b \cap a \neq \emptyset$. By example, convince me that " $=$ " is not transitive and hence " $=$ " is not an equivalence relation on B.

let $a = \{4, 6\}$, $b = \{4\}$, $c = \{6\}$

$b \cap a = \emptyset \quad \therefore \; a" = "b$ However, $b \cap c = \emptyset$ Hence $b" \neq "c$.

$c \cap a = \emptyset \quad \therefore \; a" = "c$ $\therefore$ "=" is not transitive and is not an equivalence relation on B.

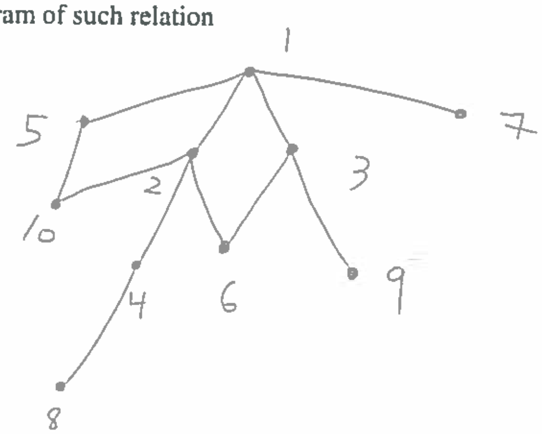
(c) Let $F = \{c \in B \mid |C| = 3\}$. Find $|F|$ (note that $|c|$ means the cardinality of c).

$7C3 = 35$

QUESTION 7. Let $A = \{1, 2, 3, ..., 9, 10\}$. Define " $\leq$ " on A such that $\forall a, b \in A$, "$a \leq b$" if and only if $a = bc$ for some $c \in A$. Then " $\leq$ " is a partial order relation on A (Do not show that).

(i) **(4 points)** Draw the Hassee diagram of such relation

1 "$\leq$" (1)
2 "$\leq$" (1, 2)
3 "$\leq$" (1, 3)
4 "$\leq$" (1, 2, 4)
5 "$\leq$" (1, 5)
6 "$\leq$" (1, 2, 3, 6)
7 "$\leq$" (1, 7)
8 "$\leq$" (1, 2, 4, 8)
9 "$\leq$" (1, 3, 9)
10 "$\leq$" (1, 2, 5, 10)



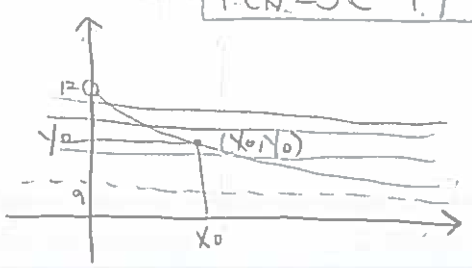
(ii) **(3 points)** By staring at the Hassee diagram, if possible, find

a. $9 \vee 6$ = 3
b. $10 \wedge 5$ = 10
c. $7 \vee 5$ = 1
d. $5 \wedge 2$ = 10
e. Is there a $c \in A$ such that $a \leq c$ for every $a \in A$? If yes, find c yes, $c = 1$
f. Is there an $m \in A$ such that $m \leq a$ for every $a \in A$? If yes, find m NO

QUESTION 8. (4 points) Convince me that $|(0, \infty)| = |[9, 12]|$ (you need to use the concept of bijective function).

$f: (0, \infty) \to (9, 12)$

$\boxed{f(x) = 3e^{-x} + 9}$



By staring, we can observe the graph is bijective. $\therefore \; |(0, \infty)| = |(9, 12)|$

Also, I need this result:

Assume $|A| = \infty$, B is countable

Then: $|A \cup B| = |A|$

Let $A = |(9, 12)|$, $B = \{9, 12\}$

$\therefore \; |(9, 12) \cup \{9, 12\}| = |[9, 12]| = |(9, 12)|$

Since cardinality is transitive, $|[9, 12]| = |(0, \infty)|$

QUESTION 9. (4 points)

(i) How many 6-digit even integers STRICTLY greater than 500002 can be formed using the digits {2, 3, 4, 5, 6, 7, 8} such that the fifth digit must be an odd integer.

{5 / 6, 7, 8} 7 7 7 3 {5 / 6 / 7} > 500002

1×7×7×7×3×4
+ 3×7×7×7×3×4
= 2 7783

1

4

4

(ii) There are 649 balls and there are 10 holes (very deep holes). The holes are labeled A, A, A, A, A B, B, B, C, C. 507 balls must be placed in A-holes (i.e., maybe all of them in one A-hole, or in two A-holes or in three A-holes or in four A-holes or in five A-holes), 33 balls must be placed in B-holes (see my earlier comment), and the remaining balls must be placed in C-Holes (again, see my earlier comment). Then there are at least n balls that are placed in the same hole (such hole could be an A-hole, a B-hole, or a C-hole). What is the maximum value of n?

$\lceil \frac{507}{5} \rceil = 102$ $\lceil \frac{109}{2} \rceil = 55$ $n = \max(102, 11, 55)$

$\lceil \frac{33}{3} \rceil = 11$ $\therefore n = 102$

QUESTION 10. (4 points)

Given $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Let f be a bijective function from S onto S such that

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 1 & 2 & 8 & 4 & 3 \end{pmatrix}$$

(i) Find f^2.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 8 & 2 & 7 & 6 & 3 & 1 & 5 \end{pmatrix}$$

(ii) Find the least positive integer n such that $f^n = I$, where I is the identity map (i.e., $I(a) = a$ for every $a \in S$)

$\underbrace{(1, 7, 4)}_{3}$ $\underbrace{(2, 6, 8, 3, 5)}_{5}$

$\text{LCM}[3, 5] = \frac{3 \times 5}{\gcd(3, 5)} = 15$ $\therefore$ the least poseteve integer es 15. $(n = 15)$

QUESTION 11. (6 points) Let $A = \{-5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6, 7, 8\}$. Define "=" on A such that $\forall a, b \in A$, $a" = "b$ if and only if $a \pmod 3 = b \pmod 3$. Then "=" is an equivalence relation. Do not show that.

(i) Find all equivalence classes of A.

$[-5] = \{-5, -2, 1, 4, 7\}$

$[-4] = \{-4, -1, 2, 5, 8\}$

$[-3] = \{-3, 3, 6\}$

(ii) view "=" as a subset of $A \times A$. How many elements does "=" have?

$5 \times 5 + 5 \times 5 + 3 \times 3 = 25 + 25 + 9 = 59$

QUESTION 12. (5 points) Let $m = gcd(28, 128)$. Then find a, b such that $m = 28a + 128b$

$128 = 4 \times 28 + 16$ (112, remainder 16)

$28 = 1 \times 16 + 12$ (16, remainder 12)

$16 = 1 \times 12 + 4$ (12, remainder 4)

$12 = 3 \times 4 + 0$ (12, remainder 0)

$gcd(28, 128) = 4$

$$4 = 16 - 12$$

$$4 = 16 - (28 - 16)$$

$$4 = (128 - 4\times 28) - \{28 - (128 - 4\times 28)\}$$

$$4 = 128 - 4\times 28 - (28 - 128 + 4\times 28)$$

$$4 = 128 - 4\times 28 - 28 + 128 - 4\times 28$$

$$\therefore 4 = -9\times 28 + 2\times 128$$

$\therefore a = -9,\ b = 2$

256

252

4

QUESTION 13. (6 points) Use math induction and convince me that $14 \mid (3^{(6m+3)} - 13)$ for every $m \geq 1$.

(i) Prove for $m = 1$

$$\frac{3^9 - 13}{14} = 1405 \qquad \therefore \text{deveseble by } 14$$

(ii) Assume $14 \mid (3^{(6m+3)} - 13)$ for some $m \geq 1$

(iii) Prove for $m+1$

Show that $14 \mid 3^{6(m+1)+3} - 13$

$$3^{6(m+1)+3} - 13 = 3^{6m+9} - 13 = 3^6 \cdot 3^{(6m+3)} - 13$$

$$= 3^6 \cdot 3^{(6m+3)} - 13 - 3^6 \cdot 13 + 3^6 \cdot 13$$

$$= 3^6 \cdot (3^{(6m+3)} - 13) + 3^6 \cdot 13 - 13$$

$$\Rightarrow \frac{3^{6(m+1)+3}}{14} = \frac{3^6 \cdot (3^{(6m+3)} - 13)}{14} + \frac{3^6 \cdot 13 - 13}{14}$$

↳ ∴ this is also divisible by 14.

↳ deveseble by (ii)

↳ deveseble by 14

$\therefore 14 \mid 3^{6m+3} - 13$ for every $m \geq 1$

QUESTION 14. (6 points) Let X be number of students in MTH 221. Given $0 < X < 63$ such that $X \ (mod\ 7) = 2$ and $3X \ (mod\ 9) = 6$. Use the Chinese remainder Theorem (CRT) and find all possible values of X.

$X \ (\text{mod } 7) = 2 \quad n_1 = 7 \qquad \boxed{m = 7 \times 9 = 63}$

$3X \ (\text{mod } 9) = 6 \quad n_2 = 9$

gcd $(7, 9) = 1$, Hence, CRT is applicable.

To find X:

$m_1 = \frac{m}{n_1} = \frac{63}{7} = \boxed{9} \qquad m_2 = \frac{m}{n_2} = \frac{63}{9} = \boxed{7}$

$9X = 1$ in Z_7 $\qquad 7X = 3$ in Z_9

$\Leftrightarrow 2X = 1$ in Z_7 $\qquad X_2 = 3$

$\therefore X_1 = 4$

1

A

$X = (m_1 X_1 C_1 + m_2 X_2 C_2) \ (\text{mod } m)$

$= ((9)(4)(2) + (7)(3)(6)) \ (\text{mod } 63)$

$= 9$

$\therefore X = \cancel{9}, \cancel{18}, \cancel{27}, \cancel{36}, \cancel{45}, \cancel{54}$

QUESTION 15. (5 points)

(1) Find all possible solution of $8x = 12$ over PLANET Z_{20}

gcd$(8, 20) = 4$ Is $4 | 12$? yes. $\therefore$ we have 4 different solutions.

To find x_1.. try and error. $8 \cdot x_1 = 12$ in Z_{20} $\quad x_1 = 4$

To find other solutions: $h = 20 = 4 \times \boxed{5}$

$\therefore$ set of other Solutions: $\{4, 9, 14, 19\}$

(2) Find all possible solution of $8x \ (mod\ 20) = 12$ over PLANET Z

$\therefore \ 4 + 5K, \ K \in Z.$

Faculty information

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